Deformations of Non-Compact, Projective Manifolds

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Abstract

In this paper, we demonstrate that the complete, hyperbolic representation of various two-bridge knot and link groups enjoy a certain local rigidity property inside of the $\mathrm{PGL}_4(\mathbb{R})$ character variety. We also prove a complementary result showing that under certain rigidity hypotheses, branched covers of amphicheiral knots admit non-trivial deformations near the complete, hyperbolic representation.

1 Introduction

Mostow rigidity for hyperbolic manifolds is a crucial tool for understanding the deformation theory of lattices in $\mathrm{Isom}(\mathbb{H}^n)$. Specifically, it tells us that the fundamental groups of hyperbolic manifolds of dimension $n \geq 3$ admit a unique conjugacy class of discrete, faithful representations into $\mathrm{Isom}(\mathbb{H}^n)$.

Recent work of [3–5, 10] has revealed several parallels between the geometry of hyperbolic n-space and the geometry of strictly convex domains in $\mathbb{RP}^n$. For example, the classification and interaction of isometries of strictly convex domains is analogous to the situation in hyperbolic geometry. Additionally, if the isometry group of the domain is sufficiently large then the strictly convex domain is known to be δ-hyperbolic. Despite the many parallels between these two types of geometry, there is no analogue of Mostow rigidity for strictly convex domains. This observation prompts the following question: is it possible to deform a finite volume, strictly convex structure on a fixed manifold?

Currently, the answer is known only in certain special cases. For example, when the manifold contains a totally geodesic, hypersurface there exist non-trivial deformations at the level of representations coming from the bending construction of Johnson and Millson [15]. In the closed case, work of Koszul [17] shows that these new projective structures arising from bending remain properly convex. Further work of Benoist [4] shows that these structures are actually strictly convex. In the non-compact case recent work of Marquis [19] has shown that the projective structures arising from bending remain properly convex in this setting as well.

In contrast to the previous results, there are examples of closed 3-manifolds for which no such deformations exist (see [9]). Additionally, there exist 3-manifolds that contain no

totally geodesic surfaces, yet still admit deformations (see [8]). Henceforth, we will refer to these deformations that do not arise from bending as *flexing deformations*.

While Mostow rigidity guarantees the uniqueness of *complete* structures on finite volume 3-manifolds, work of Thurston [25] shows that if we remove the completeness hypothesis then there is an interesting deformation theory for cusped, hyperbolic 3-manifolds. In order to obtain a complete hyperbolic structure we must insist that the holonomy of the peripheral subgroup be parabolic. Thus, if would like any sort of rigidity result for projective structures on non-compact manifolds it will be necessary to constrain how our representations restrict to the peripheral subgroups. As shown in [10], there is a notion of an automorphism of a strictly convex domain being parabolic, and in light of [10, Thm 11.6] insisting that the holonomy of the peripheral subgroup be parabolic seems to be a natural restriction.

Prompted by these results a natural question to ask is whether or not there exist flexing deformations for non-compact, finite volume manifolds. Two bridge knots and links provide a good place to begin exploring because they have particularly simple presentations for their fundamental groups. Additionally, work of [12] has shown that they contain no closed, totally geodesic, embedded surfaces. Using normal form techniques developed in section 4 we are able to prove that several two bridge knot and link complements enjoy a certain rigidity property. See section 3 for definitions related to local and infinitesimal rigidity.

Theorem 1.1. *The two bridge links with rational number $\frac{5}{2}$ (figure-eight) , $\frac{7}{3}$, $\frac{9}{5}$, and $\frac{8}{3}$ (Whitehead link) are locally projectively rigid relative to the boundary near their geometric representations.*

Remark 1.2. *The knots and links mentioned in Theorem 1.1 correspond to the 4_1, 5_2, 6_1, and 5^2_1 in Rolfsen's table of knots and links* [24]

It should be mentioned that an infinitesimal analogue of this theorem is proven in [13] for the figure-eight knot and the Whitehead link. In light of Theorem 1.1 we ask the following:

Question 1. *Is it possible to deform the geometric representation of any two-bridge knot or link relative to the boundary?*

In [13] it is shown that there is a strong relationship between deformations of a cusped hyperbolic 3-manifold and deformations of surgeries on that manifold. In particular they are able to use the fact that the figure-eight knot is infinitesimally projectively rigid relative to the boundary to deduce that there are deformations of certain orbifold surgeries of the figure-eight knot. We are able to extend these results to other amphicheiral knot complements that enjoy a certain rigidity property in the following theorem. See section 6 for the definition of rigid slope.

Theorem 1.3. *Let M be the complement of a hyperbolic, amphicheiral knot, and suppose that M is infinitesimally projectively rigid relative to the boundary and the longitude is a rigid slope. Then for sufficiently large n, $M(n/0)$ has a one dimensional deformation space.*

Here $M(n/0)$ is orbifold obtained by surgering a solid torus with longitudinal singular locus of cone angle $2\pi/n$ along the meridian of M. Other than the figure-eight knot, we cannot yet prove that there exist other knots satisfying the hypotheses of Theorem 1.3. However, there is numerical evidence that the two bridge knot with rational number $\frac{13}{5}$ satisfies these conditions. Also, in light of Theorem 1.1 and the rarity of deformations in the closed examples computed in [9], it seems that two bridge knots that are infinitesimally rigid relative to the cusps may be quite abundant. Additionally, there are infinitely many amphicheiral two-bridge knots, and so there is hope that there are many situations in which Theorem 1.3 applies.

The organization of the paper is as follows. Section 2 discusses some basics of projective geometry and projective isometries while Section 3 discusses local and infinitesimal deformations with a focus on bending and its effects on peripheral subgroups. Section 4 discusses some normal forms into which parabolic isometries can be placed. In Section 5 we use the techniques of the previous section to give a thorough discussion of deformations of the figure-eight knot (Section 5.1) and Whitehead link (Section 5.2), which along with computations in [2] proves Theorem 1.1. Finally, in Section 6 we discuss some special properties of amphicheiral knots and their representations and prove Theorem 1.3.

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2 Projective Geometry and Convex Projective Structures

Let V be a finite dimensional real vector space. We form the *projectivization of V,* denoted $P(V)$, by quotienting $V\backslash\{0\}$ by the action of $\mathbb{R}^\times$. When $V=\mathbb{R}^n$ we denote $P(\mathbb{R}^n)$ by $\mathbb{RP}^n$. If we take $\mathrm{GL}(V)$ and quotient it by the action of $\mathbb{R}^\times$ acting by multiplication by scalar matrices we get PGL(V). It is easy to verify that the action of $\mathrm{GL}(V)$ on V descends to an action of PGL(V) on $P(V)$. If $W\subset V$ is a subspace, then we call $P(W)$ a *projective subspace* of $P(V)$. Note that the codimension of $P(W)$ in $P(V)$ is the same as the codimension of W in V and the dimension of $P(W)$ is one less than the dimension of W. A *projective line* is a 1-dimensional projective subspace.

A projective space $P(V)$ admits a double cover $\pi : S(V)\to P(V)$, where $S(V)$ is the quotient of V by the action of $\mathbb{R}_+$. In the case where $V=\mathbb{R}^n$ we denote $S(V)$ by S^n. The automorphisms of $S(V)$ are $\mathrm{SL}^{\pm}(V)$, which consists of matrices with determinant ± 1.

Similarly, there is a 2-1 map from $\mathrm{SL}^{\pm}(V)$ to PGL(V).

A subset of Ω of a projective space Y is *convex* if its intersection with any projective line is connected. An *affine patch* is the complement in Y of a codimension 1 subspace. A convex subset $\Omega \subset Y$ is *properly convex* if Ω is contained in some affine patch. A point in $\partial\Omega$ is *strictly convex* if it is not contained in a line segment of positive length in $\partial\Omega$ and a Ω is *strictly convex* if it is properly convex and every point in its boundary is strictly convex. Next, we discuss how properly convex sets sit inside of S^n. Observe that if $\Omega \subset \mathbb{RP}^n$ is properly convex, then its preimage of Ω in S^n. has two components (this is easily seen by assuming that the codimension 1 subspace defining the affine patch projects to the equator of S^n). If we let $\mathrm{SL}^{\pm}(\Omega)$ and $\mathrm{PGL}(\Omega)$ be subsets of $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ and $\mathrm{PGL_{n+1}}(\mathbb{R})$ preserving Ω, respectively, then there is a nice map between $\mathrm{PGL}(\Omega)$ and $\mathrm{SL}^{\pm}(\Omega)$ that eliminates the need to deal with equivalence classes of matrices. An element $T \in \mathrm{PGL}(\Omega)$ has two lifts to $\mathrm{SL}^{\pm}(\Omega)$, one of which preserves the components of $\pi^{-1}(\Omega)$ and another which interchanges them. Sending T to the lift that preserves the components gives the desired map.

Let Ω be properly convex and open, then given a matrix $\tilde{T} \in \mathrm{SL}^{\pm}(\Omega)$ representing an element $T \in \mathrm{PGL}(\Omega)$, the fixed points of T in $\mathbb{RP}^n$ correspond to eigenvectors of $\tilde{T}$ with real eigenvalues. We can now classify elements of $\mathrm{SL}^{\pm}(\Omega)$ according to their fixed points as follows: if A fixes a point in Ω, then A is called *elliptic*. If A acts freely on Ω, and all of its eigenvalues have modulus 1 then A is *parabolic*. Otherwise A is *hyperbolic*. It is shown in [10] that parabolic elements of $\mathrm{SL}^{\pm}(\Omega)$ are subject to certain linear algebraic constraints.

Theorem 2.1. *Suppose that Ω is a properly convex domain and that $A \in \mathrm{SL}^{\pm}(\Omega)$ is parabolic. Then one of largest Jordan blocks of A has eigenvalue 1. Additionally, the size of this Jordan block is odd and at least 3. If Ω is strictly convex then this is the only Jordan block of this size.*

This theorem is particularly useful in small dimensions. For example, in dimensions 2 and 3 any parabolic that preserves a properly convex subset is conjugate to

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. This theorem also tells us that in these small dimensional cases, 1 is the only possible eigenvalue of a parabolic that preserves a properly convex domain.

When Ω is properly convex we can also gain more insight into the structure of $\mathrm{SL}^{\pm}(\Omega)$ by putting an invariant metric on Ω. Given $x_1, x_2 \in \Omega$ we define the *Hilbert metric* as follows: let ℓ be the line segment between x_1 and x_2. Proper convexity tells us that ℓ intersects the boundary in two points y and z, and we define $d_{\mathrm{H}}(x_1, x_2) = |\log([y : x_1 : x_2 : z])|$, where $[y : x_1 : x_2 : z]$ is the cross ratio, defines a metric on Ω. This metric is clearly invariant under $\mathrm{SL}^{\pm}(\Omega)$ because the cross ratio is invariant under projective automorphisms. This metric gives rise to a Finsler metric on Ω, and in the case that Ω is an ellipsoid it coincides with twice the standard hyperbolic metric.

We now use this metric to understand certain subsets of $\mathrm{SL}^{\pm}(\Omega)$.

Lemma 2.2. *Let Ω be a properly convex domain and let x be a point in the interior of Ω, then the set $\Omega_x^K = \{T \in \mathrm{SL}^{\pm}(\Omega) \mid d_{\mathrm{H}}(x, Tx) \leq K\}$ is compact.*

Proof. Let $\mathcal{B} = \{x_0, \dots, x_n)\}$ be a projective basis (that is a set of $n+1$ point no n of which live in a common hyperplane) that are contained in Ω and such that $x_0 = x$. Next, let γ_i be a sequence of elements of Ω_x^K. the elements $\gamma_i x_0$ all live in the compact ball of radius K centered at x and so by passing to subsequence we can assume that $\gamma_i x_0 \to x_0^\infty \in \Omega$. Next, we claim that the by passing to subsequence that $\gamma_i x_j \to x_j^\infty \in \Omega$ for $1 \leq j \leq n$. To see this observe that

$$d_{\mathrm{H}}(x_0^\infty, \gamma_i x_j) \leq d_{\mathrm{H}}(x_0^\infty, \gamma_i x_0) + d_{\mathrm{H}}(\gamma_i x_0, \gamma_i x_j) = d_{\mathrm{H}}(x_0^\infty, \gamma_i x_0) + d_{\mathrm{H}}(x_0, x_j),$$

and so all of the $\gamma_i x_j$ live in a compact ball centered at x_0^∞. Elements of$\mathrm{PGL_{n+1}}(\mathbb{R})$ are uniquely determined by their action on a projective basis, and convergence of elements of $\mathrm{PGL_{n+1}}(\mathbb{R})$ is the same as convergence bases as subsets of $(\mathbb{RP}^n)^{n+1}$. Therefore the proof will be complete if we can show that the set $\{x_0^\infty \dots, x_n^\infty\}$ is a projective basis. Suppose that this set is not a basis, then without loss of generality we can assume that the set $\{v_0^\infty, \dots, v_{n-1}^\infty\}$ is linearly dependent, where $x_i = [v_i]$. Thus $v_0^\infty = c_1 v_1^\infty + \dots + c_{n-1} v_{n-1}^\infty$ is a non-trivial linear combination, and we find that $\gamma_i [c_1 v_1 + \dots + c_{n-1} v_{n-1}] \to [v_0^\infty]$. However,

$$d_{\mathrm{H}}([v_0], [c_1 v_1 + \dots + c_{n-1} v_{n-1}]) = d_{\mathrm{H}}(\gamma_i [v_0], \gamma_i [c_1 v_1 + \dots + c_{n-1} v_{n-1}])$$

and

$$d_{\mathrm{H}}(\gamma_i [v_0], \gamma_i [c_1 v_1 + \dots + c_{n-1} v_{n-1}]) \to d_{\mathrm{H}}([v_0^\infty], [c_1 v_1^\infty + \dots + c_{n-1} v_{n-1}^\infty]) = 0,$$

which contradicts the fact that $\mathcal{B}$ is a basis. □

As we mentioned previously, isometries of a strictly convex domain interact in ways similar to hyperbolic isometries. As an example of this, recall that if $\phi, \psi \in \mathrm{SO}(n,1)$, with ϕ hyperbolic, then ϕ and ψ cannot generate a discrete group if they share exactly one fixed point. In particular parabolics and hyperbolics cannot share fixed points in a discrete group. A similar phenomenon takes place inside of $\mathrm{SL}^{\pm}(\Omega)$, when Ω is strictly convex.

Proposition 2.3. *Let Ω be a strictly convex domain and $\phi, \psi \in \mathrm{SL}^{\pm}(\Omega)$ with ϕ hyperbolic. If ϕ and ψ have exactly one fixed point in common, then the subgroup generated by ϕ and ψ is not discrete.*

Proof. Notice the similarity between this proof and the proof in [22, Thm 5.5.4]. Suppose for contradiction that the subgroup generated by ϕ and ψ is discrete. Since Ω is strictly convex, ϕ has exactly two fixed points (see [10, Prop 2.8]), x_1 and x_2 and without loss of generality we can assume that they correspond to the eigenvalues of smallest and largest modulus, respectively, and that ψ fixes x_1 but not x_2. From [10, Prop 4.6] we know that x_1

is a C^1 point, and hence there is a unique supporting hyperplane to Ω at x_1 and both ϕ and ψ preserve it. This means that there are coordinates with respect to which both ϕ and ψ are affine. In particular we can assume that

$$\phi(x) = Ax, \ \psi(x) = Bx + c,$$

with $c \neq 0$. We now examine the result of conjugating ψ by powers of ϕ.

$$\phi^n \psi \phi^{-n}(x) = A^n B A^{-n} + A^n c$$

The fixed point x_2 (which had the largest eigenvalue) has now been moved to the origin and has been projectively scaled to have eigenvalue 1. Since Ω is strictly convex, x_2 is the unique attracting fixed point, and so after possibly passing to a subsequence we can assume that $\{A^n c\}$ is a sequence of distinct vectors that converge to the origin. Since $A^n c = \phi^n \psi \phi^{-n}(0)$ we see that $\{\phi^n \psi \phi^{-n}\}$ is also a distinct sequence. The elements of $\{\phi^n \psi \phi^{-n}\}$ all move points on line between the fixed points of ϕ a fixed bounded distance, and so by Lemma 2.2 this sequence has a convergent subsequence, which by construction, is not eventually constant. The existence of such a sequence contradicts discreteness.

$\square$

To close this section we briefly describe convex real projective structures on manifolds. For more details about real projective structures and more generally (G, X) structures see [22, 25]. A *real projective structure* on an n-manifold M is an atlas of charts $U \to \mathbb{RP}^n$ such that the transition functions live in $\mathrm{PGL_n}(\mathbb{R})$. We can globalize the data of an atlas by selecting a chart and constructing a local diffeomorphism $D : \tilde{M} \to \mathbb{RP}^n$ using analytic continuation. This construction also yields a representation $\rho : \pi_1(M) \to \mathrm{PGL_n}(\mathbb{R})$ that is equivariant with respect to D. The map and representation are known as a *developing map* and *holonomy representation*, respectively. Only the initial choice of chart results in ambiguity in the developing construction, and different choices of initial charts will result in developing maps which differ by post composing by an element of $g \in \mathrm{PGL_n}(\mathbb{R})$. Additionally, the holonomy representations will differ by conjugation by g.

When the map D is a diffeomorphism onto a properly (resp. strictly) convex set we say that the real projective structure is *properly (resp. strictly) convex*. A key example to keep in mind are hyperbolic structures. Projectivizing the cone $x_1^2 + \ldots x_n^2 < x_{n+1}^2$ gives rise to the Klein model of hyperbolic space. The Klein model can be embedded in affine space as the unit disk and so we realize hyperbolic structures as specific instances of strictly convex real projective structures.

3 Local and Infinitesimal Deformations

Unless explicitly mentioned, Γ will henceforth denote the fundamental group of a complete, finite volume, hyperbolic 3-manifold. By Mostow rigidity, there is a unique class of representations of Γ that is faithful and has discrete image in $\mathrm{SO}(3, 1)$. We call this class the

geometric representation and denote it $[\rho_{\mathrm{geo}}]$. From the previous section we know that real projective structures on a manifold give rise to conjugacy classes of representations of its fundamental group into $\mathrm{PGL}_{\mathrm{n}}(\mathbb{R})$. Therefore, if we want to study real projective structures near the complete hyperbolic structure it suffices to study conjugacy classes of representations near $[\rho_{\mathrm{geo}}]$.

We now set some notation. Let $\mathcal{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ be the $\mathrm{PGL}_4(\mathbb{R})$ *representation variety* of Γ. The group $\mathrm{PGL}_4(\mathbb{R})$ acts on $\mathcal{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ by conjugation, and the quotient by this action is the *character variety*, which we denote $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$. The character variety is not globally a variety because of pathologies of the action by conjugation, however at ρ_{geo} the action is nice enough to guarantee that $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ has the local structure of a variety.

Next, we define a refinement of $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ that better controls the representations on the boundary. Recall, that by Theorem 2.1 then only conjugacy class of parabolic element that is capable of preserving a properly convex domain is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Because this is the conjugacy class of a parabolic element of $\mathrm{SO}(3,1)$ we will refer to parabolics of this type as $\mathrm{SO}(3,1)$ *parabolics*. Let $\mathcal{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ be the elements of $\mathcal{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ such that peripheral elements of Γ are mapped to $\mathrm{SO}(3,1)$ parabolics, and let the *relative character variety*, denoted $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$, be the corresponding quotient by the action $\mathrm{PGL}_4(\mathbb{R})$.

If ρ is a representation, then a *deformation of ρ* is a smooth map, $\sigma(t) : (-\varepsilon, \varepsilon) \to \mathcal{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ such that $\sigma(0) = \rho$. Often times we will denote $\sigma(t)$ by σ_t. If a class $[\sigma]$ is an isolated point of the $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ then we say that Γ is *locally projectively rigid at σ*. Similarly, we say that $[\sigma]$ is *locally rigid relative relative to the boundary* when it is isolated in $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$.

3.1 Twisted Cohomology and Infinitesimal Deformations

We now review how the cohomology of Γ with a certain system of local coefficients helps to infinitesimally parameterize conjugacy classes of deformations of representations. For more details on cohomology see [7, Chap III]

Let $\sigma_0 : \Gamma \to \mathrm{PGL}_4(\mathbb{R})$ be a representation, then we can define a cochain complex, $C^n(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$ as function from Γ^n to $\mathfrak{sl}(4)$, with differential $d^n : C^n(\Gamma, \mathfrak{sl}(4)_{\sigma_0}) \to C^{n+1}(\Gamma, \mathfrak{sl}(4)_{\rho_0})$ where $d^n\phi(\gamma_1, \ldots, \gamma_{n+1})$ is given by

$$\gamma_1 \cdot \phi(\gamma_2, \ldots, \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \ldots, \gamma_{n+1}) + (-1)^{n+1}\phi(\gamma_1, \ldots, \gamma_n),$$

where $\gamma \in \Gamma$ acts on $\mathfrak{sl}(4)$ by $\mathrm{Ad}(\sigma_0(\gamma))$. Letting $Z^n(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$ and $B^n(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$ be the kernel of d^n and image of d^{n-1}, respectively, we can form the cohomology groups $H^*(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$.

To see how this construction is related to deformations, let σ_t be a deformation of σ_0, then since $\mathrm{PGL}_4(\mathbb{R})$ is a Lie group we can use a series expansion and write $\sigma_t(\gamma)$ as

$$\sigma_t(\gamma) = (I + z(\gamma)t + O(t^2))\sigma_0(\gamma), \tag{3.1}$$

where $\gamma \in \Gamma$ and z is a map from Γ into $\mathfrak{sl}(4)$, which we call an *infinitesimal deformation*. The above construction would have worked for any smooth function from $\mathbb{R}$ to $\mathrm{PGL}_4(\mathbb{R})$, but the fact that σ_t is a homomorphism for each t puts strong restrictions on z. Let γ and γ' be elements of Γ, then

$$\sigma_t(\gamma\gamma') = (I + z(\gamma\gamma')t + O(t^2))\sigma_0(\gamma\gamma') \tag{3.2}$$

and

$$\begin{aligned}\sigma_t(\gamma)\sigma_t(\gamma') &= (I + z(\gamma)t + O(t^2))\sigma_0(\gamma)(I + z(\gamma') + O(t^2))\sigma_0(\gamma') \\ &= (I + (z(\gamma) + \gamma \cdot z(\gamma'))t + O(t^2))\sigma_0(\gamma\gamma'),\end{aligned}$$

where the action is the adjoint action of σ_0, given by $\gamma \cdot M = \sigma_0(\gamma)M\sigma_0(\gamma)^{-1}$. By focusing on the linear terms of the two power series in (3.2), we find that $z(\gamma\gamma') = z(\gamma) + \gamma \cdot z(\gamma')$, and so $z \in Z^1(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$, the set of 1-cocycles twisted by the action of σ_0. Since we think of deformations coming from conjugation as uninteresting, we now analyze which elements in $Z^1(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$ arise from these types of deformations. Let c_t be a smooth curve in $\mathfrak{sl}(4)$ such that $c_0 = I$, and consider the deformation $\sigma_t(\gamma) = c_t^{-1}\sigma_0(\gamma)c_t$. Again we write $\sigma_t(\gamma)$ as a power series we find that

$$c_t^{-1}\sigma_0(\gamma)c_t = (I - z_c t + O(t^2))\sigma_0(\gamma)(I + z_c t + O(t^2)) = (I + (z_c - \gamma \cdot z_c)t + O(t^2))\sigma_0(\gamma), \tag{3.3}$$

and again by looking at linear terms we learn $z(\gamma) = z_c - \gamma \cdot z_c$ and so z is a 1-coboundary.

We therefore conclude that studying infinitesimal deformations near σ_0 up to conjugacy boils down to studying $H^1(\Gamma, \mathfrak{sl}(4)_{\sigma_0})$. In the case where $H^1(\Gamma, \mathfrak{sl}(4)_{\sigma_0}) = 0$ we say that Γ is *infinitesimally projectively rigid at* σ_0, and when the map $\iota^* : H^1(\Gamma, \mathfrak{sl}(4)_{\sigma_0}) \to H^1(\pi_1(M), \mathfrak{sl}(4)_{\sigma_0})$ induce by the inclusion $\iota : \partial M \to M$ is injective we say that Γ is *infinitesimally projectively rigid relative to the boundary at* σ_0. The following theorem of Weil [26] shows the strong relationship between infinitesimal and local rigidity.

Theorem 3.1. *If Γ is infinitesimally projectively rigid at σ then Γ is locally projectively rigid at σ.*

More generally the dimension of $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_{\mathrm{geo}}})$ is an upper bound for the dimension of $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ at $[\rho_{\mathrm{geo}}]$ (see [15, Sec 2]). However, it is important to remember that in general the character variety need not be smooth. When this occurs this bound is not sharp and so the converse to Theorem 3.1 is false.

3.2 Decomposing $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_0})$

In order to simplify the study of $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_0})$ we will decompose it into two factors using a decomposition of $\mathfrak{sl}(4)$. Consider the symmetric matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Following [13, 15] we see that this gives rise to the following decomposition of $\mathrm{PSO}(3,1)$ modules.

$$\mathfrak{sl}(4) = \mathfrak{so}(3,1) \oplus \mathfrak{v}, \tag{3.4}$$

where $\mathfrak{so}(3,1) = \{a \in \mathfrak{sl}(4) \mid a^t J = -Ja\}$ and $v = \{a \in \mathfrak{sl}(4) \mid a^t J = Ja\}$. These two spaces can be thought of as the ± 1-eigenspaces of the involution $a \mapsto -Ja^t J$. This splitting of $\mathfrak{sl}(4)$ gives rise to a splitting of the cohomology groups, namely

$$H^*(\Gamma, \mathfrak{sl}(4)_{\rho_0}) = H^*(\Gamma, \mathfrak{so}(3,1)_{\rho_0}) \oplus H^*(\Gamma, \mathfrak{v}_{\rho_0}). \tag{3.5}$$

If ρ_{geo} is the geometric representation then the first factor of (3.5) is well understood. By work of Garland [11], $H^1(M, \mathfrak{so}(3,1)_{\rho_{\mathrm{geo}}})$ injects into $H^1(\partial M, \mathfrak{so}(3,1)_{\rho_{\mathrm{geo}}})$ and work of Thurston [25] shows that $\dim H^1(M, \mathfrak{so}(3,1)_{\rho_{\mathrm{geo}}}) = 2k$, where k is the number of cusps of M.

Now that we understand the structure of $H^1(\Gamma, \mathfrak{so}(3,1)_{\rho_0})$ we turn our attention to the other factor of (3.5). The inclusion $\iota : \partial M \to M$ induces a map $\iota^* : H^1(M, \mathfrak{sl}(4)_{\rho_{\mathrm{geo}}}) \to H^1(\partial M, \mathfrak{sl}(4)_{\rho_{\mathrm{geo}}})$ on cohomology. We will refer to the kernel of this map as the $\mathfrak{sl}(4)_{\rho_{\mathrm{geo}}}$-*cuspidal cohomology*. In [13], Porti and Heusener analyze the image of this map. The portion that we will use can be summarized by the following theorem, which can be thought of as a twisted cohomology analogue of the classical half lives/half dies theorem.

Theorem 3.2. *Let ρ_{geo} be the geometric representation of a finite volume hyperbolic 3-manifold with k cusps, then $\dim(\iota^*(\mathrm{H}^1(\Gamma, \mathfrak{v}_{\rho_{\mathrm{geo}}})))$=$k$. Furthermore, if $\partial M = \sqcup_{i=1}^k \partial M_i$, then there exists $\gamma = \sqcup_{i=1}^k \gamma_i$ with $\gamma_i \subset \partial M_i$, such that $\iota^*(H^1(\Gamma, \mathfrak{v}_{\rho_{\mathrm{geo}}}))$ injects into $\bigoplus_{i=1}^k H^1(\gamma_i, \mathfrak{v}_{\rho_{\mathrm{geo}}})$*

3.3 Bending

Now that we have some upper bounds on the dimensions of our deformations spaces we want to begin to understand what types of deformations exist near the discrete faithful representation. The most well know construction of such deformations is bending along a totally geodesic hypersurface. The goal of this section is to explain the bending construction and prove the following theorem on the effects of bending on the peripheral subgroups

Theorem 3.3. *Let S be a totally geodesic hypersurface and let $[\rho_t] \in \mathfrak{R}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$, obtained by bending along S. Then $[\rho_t]$ is contained in $\mathfrak{R}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))_{\mathrm{p}}$ if and only if S is closed or each curve dual to the intersection of ∂M and S has zero signed intersection with S.*

Following [15], let M be a finite volume hyperbolic manifold of dimension ≥ 3 and ρ_{geo} be its geometric representation. Next, let S be a properly embedded, totally geodesic, hypersurface. Such a hypersurface gives rise to a curve of representations in $\mathfrak{R}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ passing through ρ_0 as follows. We begin with a lemma showing that the hypersurface S gives rise to an element of $\mathfrak{sl}(n+1)$ that is invariant under $\Delta = \pi_1(S)$ but not all of $\Gamma = \pi_1(\Gamma)$.

Lemma 3.4. *Let M and S as above, then there exists a unique 1-dimensional subspace of $\mathfrak{sl}(n+1)$ that is invariant under Δ. Furthermore this subspace is generated by a conjugate in $PGL_{n+1}(\mathbb{R})$ of*

$$\begin{pmatrix} -n & 0 \\ 0 & I \end{pmatrix},$$

where I is the $n \times n$ identity matrix.

Proof. Γ is a subgroup of $PO(n,1)$ (the projective orthogonal group of the form $x_1^2 + x_2^2 + \ldots x_n^2 - x_{n+1}^2$). Since S is totally geodesic we can assume that after conjugation that Δ preserves both the hyperplane where $x_1 = 0$ and its orthogonal complement which is generated by $(1, 0, \ldots, 0)$. Hence if $A \in \Delta$ then

$$A = \left(\begin{array}{cc} 1 & 0^T \\ 0 & \tilde{A} \end{array} \right),$$

where $\tilde{A} \in PO(n-1,1)$ (the projective orthogonal group of the form $x_2^2 + x_3^2 + \ldots + x_n^2 - x_{n+1}^2$) and $0 \in \mathbb{R}^n$. If $x \in \mathfrak{sl}(n+1)$ is invariant under Δ then we know that $B(t) = \exp(tx)$ commutes with every $A \in \Delta$. If we write $B(t) = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right)$, where $b \in \mathbb{R}$, $b_{12}^T, b_{21} \in \mathbb{R}^n$, and $b_{22} \in \mathrm{SL_n}(\mathbb{R})$, then

$$\left(\begin{array}{cc} b_{11} & b_{12} \\ \tilde{A}b_{21} & \tilde{A}b_{22} \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & \tilde{A} \end{array} \right) \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) \left(\begin{array}{cc} 1 & 0 \\ 0 & \tilde{A} \end{array} \right) = \left(\begin{array}{cc} b_{11} & b_{12}\tilde{A} \\ b_{21} & b_{22}\tilde{A} \end{array} \right).$$

From this computation we learn that b_{12} and b_{21} are invariants of Δ and that b_{22} is in the commutator of Δ in $SL_n(\mathbb{R})$. However, the representation of Δ into $PO(n-1,1)$ is irreducible, and so the only matrices that commute with every element of Δ are scalar matrices and the only invariant vector of Δ is 0, and so

$$B = \left(\begin{array}{cc} e^{-n\lambda t} & 0 \\ 0 & e^{\lambda t} I \end{array} \right),$$

where I is the identity matrix. Differentiating $B(t)$ at $t = 0$ we find that

$$x = \left(\begin{array}{cc} -n\lambda & 0 \\ 0 & \lambda I \end{array} \right),$$

and the result follows.

$\square$

The vector from Lemma 3.4 will be called a *bending cocycle*, and we will denote it x_S. We can now define a family of deformations of ρ. The construction breaks into two cases depending on whether or not S is separating.

If S is separating then Γ splits as the following amalgamated free product:

$$\Gamma \cong \Gamma_1 *_\Delta \Gamma_2,$$

where Γ_i are the fundamental groups of the components of the complement of S in M, and we can define a family of representations ρ_t as follows. Since ρ is irreducible we know that x_S is not invariant under all of Γ and so we can assume without loss of generality that it is not invariant under Γ_2. So let $\rho_t|_{\Gamma_1} = \rho$ and $\rho_t|_{\Gamma_2} = \mathrm{Ad}(\exp(tx_S)) \cdot \rho$. Since these two maps agree on Δ they give a well defined family of homomorphisms of Γ, such that $\rho_0 = \rho$.

If S is nonseparating, then Γ is realized as the following HNN extension:

$$\Gamma \cong \Gamma' *_\Delta,$$

where Γ' is the fundamental group of $M \backslash S$. If we let α be a curve dual to S then we can define a family of homomorphisms through ρ by $\rho_t|_{\Gamma'} = \rho$ and $\rho_t(\alpha) = \exp(tx_S)\rho(\alpha)$. Since x_S is invariant under Δ the values of $\rho_t(\iota_1(\Delta))$ do not depend on t, where ι_1 is the inclusion of the positive boundary component of a regular neighborhood of S into $M \backslash S$, and so we have well defined homomorphisms of the HNN extension.

In both cases ρ_t gives rise to a non-trivial curve of representations and by examining the class of $\rho_t \in H^1(\Gamma, \mathfrak{sl}(n+1)_{\rho_0})$ Johnson and Millson [15] showed that $[\rho_t]$ actually defines a non-trivial path in $\mathfrak{R}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$.

We can now prove Theorem 3.3.

Proof of Theorem 3.3. In the case that S is closed and separating a peripheral element, $\gamma \in \Gamma$, is contained in either Γ_1 or Γ_2 since it is disjoint from S. In this case $\rho_t(\gamma)$ is either $\rho(\gamma)$ or some conjugate of $\rho(\gamma)$. In either case we have not changed the conjugacy class of any peripheral elements and so $[\rho_t]$ is a curve of representations in $\mathcal{R}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))_\mathrm{p}$. Similarly if S is closed and non-separating then if $\gamma \in \Gamma$ is peripheral, then $\gamma \in \pi_1(M \backslash S)$, and so its conjugacy class does not depend on t.

In the case that S is non-compact we need to analyze its intersection with the boundary of M more carefully. First, let M' be a finite sheeted cover of M and S' a lift of S in M'. If bending along S preserves the peripheral structure of M then bending along S' will preserve the peripheral structure of M'. Therefore, without loss of generality we may pass to a finite sheeted cover of M such that $\partial M' = \sqcup_{i=1}^k T_i$, where each of the T_i is a torus (this is possible by work of [20]). By looking in the universal cover, it is easy to see that since S is properly embedded and totally geodesic that for any boundary component T_i, $T_i \cap S = \sqcup_{j=1}^l t_j$, where the t_j are parallel $n-2$ dimensional tori. The result of the bending deformation on the boundary will be to simultaneously bend along all of these parallel tori, however some of the parallel tori may bend in opposite directions and can sometimes cancel with one another. In fact, if we look at a curve $\alpha \in T_i$ dual to one (hence all) of the parallel

tori, then intersection points of α with S are in bijective correspondence with the t_j, and the direction of the bending corresponds to the signed intersection number of α with S.

When S is separating the signed intersection number of α with S is always zero, and so bending along S has no effect on the boundary. When S is non-separating then the signed intersection can be either zero of non-zero. In the case where the intersection number is non-zero the boundary becomes non-parabolic after bending. This can be seen easily by looking at the eigenvalues of peripheral elements after they have been bent.

□

Remark 3.5. *The Whitehead link is an example where bending along a non-separating totally geodesic surface has no effect on the peripheral subgroup of one of the components (See section 5.2)*

Corollary 3.6. *Let M be a finite volume, hyperbolic 3-manifold. If M is locally projectively rigid relative the boundary near the geometric representation then M contains no closed, embedded totally geodesic surfaces or finite volume, embedded, separating, totally geodesic surfaces.*

The Menasco-Reid conjecture [21] asserts that hyperbolic knot complements do not contain closed, totally geodesic surfaces. As a consequence of Corollary 3.6 we see that any hyperbolic knot complement that is locally projectively rigid relative to the boundary near the geometric representation will satisfy the conclusion of the Menasco-Reid conjecture.

4 Some Normal Forms

The goal of this section will be to examine various normal forms into which we can put two non commuting $\mathrm{SO}(3,1)$ parabolics. In [23], Riley shows how two non commuting parabolic elements a and b inside of $\mathrm{SL}_2(\mathbb{C})$ can be simultaneously conjugated into the following form:

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}. \tag{4.1}$$

In this same spirit we would like to take two $\mathrm{SO}(3,1)$ parabolics A and B that are sufficiently close to the $\mathrm{SO}(3,1)$ parabolics A_0 and B_0 in $\mathrm{SO}(3,1)$ and show that A and B can be simultaneously conjugated into the following normal form, similar to (4.1).

$$A = \begin{pmatrix} 1 & 0 & 2 & 1+a_{14} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31}+b_{21}b_{32} & 2b_{32} & 1 & 0 \\ b_{21}+b_{41} & 2 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

At first, this may appear to be an odd normal for, however, it provides a nice symmetry between A and B and their inverses. To start we need to discuss how to build homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{PGL}_4(\mathbb{R})$ that takes our old favorite normal forms from $\mathrm{SL}_2(\mathbb{C})$ to our new

favorite normal forms in $\mathrm{PGL}_4(\mathbb{R})$. More precisely if a and b are two parabolics in $\mathrm{PSL}_2(\mathbb{C})$, then our homomorphism should take a and b to matrices of the form (4.2)

Let

$$X = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{pmatrix}$$

be a symmetric matrix. If $\det X < 0$, then X has signature $(3,1)$ (up to $\pm I$). We want to discover the restrictions placed on the coefficients of A, B, and X by knowing that the quadratic form determined by X is preserved by matrices of type (4.2). The restriction that $A^T XA = X$ and $B^T XB = X$ tell us that

$$a_{14} = -2b_{32},\ b_{31} = 0,\ b_{41} = -2,\ x_{12} = -1,\ x_{13} = 2b_{32},\ x_{14} = -b_{21}/2, \tag{4.3}$$
$$x_{22} = 1,\ x_{23} = -2b_{32},\ x_{24} = 2b_{32}^2,\ x_{33} = b_{21},\ x_{34} = -b_{21}b_{32},\ x_{44} = b_{21}b_{32}^2$$

With these restrictions we see that our matrix X looks like

$$X = \begin{pmatrix} 1 & -1 & 2b_{32} & -b_{21}/2 \\ -1 & 1 & -2b_{32} & 2b_{32}^2 \\ 2b_{32} & -2b_{32} & b_{21} & -b_{21}b_{32} \\ -b_{21}/2 & 2b_{32}^2 & -b_{21}b_{32} & b_{21}b_{32}^2 \end{pmatrix}$$

For the time being we will assume that the entries of A and B satisfy (4.3).

If we let x, y, z, t be coordinates for $\mathbb{R}^4$ then we see that the quadratic form on $\mathbb{R}^4$ by X is

$$x^2 - 2xy + y^2 - b_{21}xt + 4b_{32}^2 yt - 2b_{21}b_{32}zt + 4b_{32}xz - 4b_{32}yz + b_{21}z^2 + b_{21}b_{32}^2 t^2 \tag{4.4}$$

Let $d = b_{21} - 4b_{32}^2$, then a simple calculation shows that $\det X < 0$ if and only if $d > 0$.

Next, consider the map of quadratic spaces from $(\mathbb{R}^4, X)$ to $(\mathrm{Herm}_2, -Det)$, where Herm_2 is the space of 2×2 hermitian matrices, that takes (x, y, z, t) to

$$\begin{pmatrix} x & x - y + 2b_{32}z - 2b_{32}^2 t + i(\sqrt{d}z - b32\sqrt{d}t) \\ x - y + 2b_{32}z - 2b_{32}^2 t - i(\sqrt{d}z - b32\sqrt{d}t) & dt \end{pmatrix} \tag{4.5}$$

Let M be a 2×2 matrix and let $N \in \mathrm{SL}_2(\mathbb{C})$ act on M by $N \cdot M = NMN^*$, where $*$ denotes the conjugate transpose operator on matrices. This action is linear and clearly preserves determinants and therefore gives us a map, ϕ', from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SO}(3,1)$. Since $-I$ acts trivially this map descends to another map that we also call ϕ' from $\mathrm{PSL}_2(\mathbb{C})$ to $\mathrm{SO}(3,1)$. A simple calculation shows that

$$\phi'\left(\begin{pmatrix} 1 & i/\sqrt{d} \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 2 & 1 - 2b_{32} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $g \in \mathrm{PSL}_2(\mathbb{C})$ be a matrix that fixes 0 and ∞ and sends $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & i/\sqrt{d} \\ 0 & 1 \end{pmatrix}$, then if we let $\phi = \phi' \circ c_g$, where c_g is conjugation by g, then

$$\phi\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)\right) = \left(\begin{array}{cccc} 1 & 0 & 2 & 1-2b_{32} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

A simple, yet tedious, computation shows that if

$$b_{21} = |\omega|^2 \text{ and } b_{32} = \mathrm{Re}(\omega)/2 \tag{4.6}$$

then ϕ will take elements of the form (4.1) to elements of the form (4.2). With these assumptions on b_{21} and b_{32} we see that

$$d = b_{21} - 4b_{32}^2 = |\omega|^2 - \mathrm{Re}(\omega)^2 = \mathrm{Im}(\omega)^2,$$

and so as long as $\mathrm{Im}(\omega) \neq 0$, $\phi(a)$ and $\phi(b)$ will preserve a form of signature $(3,1)$. One of the utilities of the previous computation is that it can help us identify the discrete faithful representation in $\mathrm{PGL}_4(\mathbb{R})$ in the normal form (4.2), provided that we have found the discrete faithful representation in the normal form (4.1) inside of $\mathrm{SL}_2(\mathbb{C})$. To see this observe that if we have the discrete, faithful representation in the form (4.1), then combining the value of ω with equations (4.3) and (4.6) we can construct the entries of our matrices in the normal form (4.2).

Let $F_2 = \langle \alpha, \beta \rangle$ be the free group on two letters and let $\mathcal{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ be the set of homomorphisms, f, from F_2 to $\mathrm{PGL}_4(\mathbb{R})$ such that $f(\alpha)$ and $f(\beta)$ are $\mathrm{SO}(3,1)$ parabolics. The topology of $\mathcal{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ is induced by convergence of the generators. There is a natural action of $\mathrm{PGL}_4(\mathbb{R})$ on $\mathcal{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ by conjugation, and we denote the quotient of this action by $\mathfrak{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$. We have the following lemma about the local structure of $\mathfrak{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$.

Lemma 4.1. *Let $f_0 \in \mathcal{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ satisfy the following conditions*

1. *$\langle f_0(\alpha), f_0(\beta) \rangle$ is irreducible and conjugate into* $\mathrm{SO}(3,1)$.
2. *$\langle f_0(\alpha), f_0(\beta) \rangle$ is not conjugate into* $\mathrm{SO}(2,1)$.

Then for $f \in \mathcal{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ sufficiently close to f_0 there exists a unique (up to $\pm I$) element $G \in \mathrm{SL}_4(\mathbb{R})$ such that

$$G^{-1}f(\alpha)G = \left(\begin{array}{cccc} 1 & 0 & 2 & 1+a_{14} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad G^{-1}f(\beta)G = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31}+b_{21}b_{32} & 2b_{32} & 1 & 0 \\ b_{21}+b_{41} & 2 & 0 & 1 \end{array}\right)$$

Additionally, the map taking f to its normal form is continuous.

Proof. The previous argument combined with properties 1 and 2 ensure that $f_0(\alpha)$ and $f_0(\beta)$ can be put into the form (4.2). Let $A = f(\alpha)$ and $B = f(\beta)$. Let E_A and E_B be the 1-eigenspaces of A and B, respectively. Since both A and B are $\mathrm{SO}(3,1)$ parabolics both of these spaces are 2-dimensional. Irreducibility is an open condition we can assume that f is also irreducible and so E_A and E_B have trivial intersection. Therefore $\mathbb{R}^4 = E_A \oplus E_B$. If we select a basis with respect to this decomposition then our matrices will be of the following block form.

$$\begin{pmatrix} I & A_U \\ 0 & A_L \end{pmatrix}, \qquad \begin{pmatrix} B_U & 0 \\ B_L & I \end{pmatrix}.$$

Observe that 1 is the only eigenvalue of A_L (resp. B_U) and that neither of these matrices is diagonalizable (otherwise $(A-I)^2 = 0$ (resp. $(B-I)^2 = 0$) and so A (resp. B) would not have the right Jordan form). Thus we can further conjugate E_A and E_B so that

$$A_L = \begin{pmatrix} 1 & a_{34} \\ 0 & 1 \end{pmatrix}, \qquad B_U = \begin{pmatrix} 1 & 0 \\ b_{21} & 1 \end{pmatrix},$$

where $a_{34} \neq 0 \neq b_{21}$. Conjugacies that preserve this form are all of the form

$$\begin{pmatrix} u_{11} & 0 & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

Finally, a tedious computation[1] allows us to determine that there exist unique values of the u_{ij}s that will finish putting our matrices in the desired normal form. Note that the existence of solutions depends on the fact that the entries of A and B are close to the entries of $f_0(\alpha)$ and $f_0(\beta)$, which live in $\mathrm{SO}(3,1)$. ☐

Lemma 4.1 gives us the following information about the dimension of $\mathfrak{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$.

Corollary 4.2. *The space $\mathfrak{R}(F_2, \mathrm{PGL}_4(\mathbb{R}))_{\mathrm{p}}$ is 5-dimensional.*

We conclude this section by introducing another normal form for $\mathrm{SO}(3,1)$ parabolics. If we begin with A and B in form (4.2) then we can conjugate by the matrix

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2-b_{21}-b_{41}}{4} & \frac{2+b_{21}+b_{41}}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{2b_{32}+b_{21}b_{32}+b_{32}b_{41}-1}{2} \\ 0 & 0 & 0 & \frac{2+b_{21}+b_{41}}{2} \end{pmatrix}$$

and then change variables, the resulting form will be

$$A = \begin{pmatrix} 1 & 0 & 1 & a_{14} \\ 0 & 1 & 1 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31} & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{4.7}$$

[1]This computation is greatly expedited by using Mathematica.

Conjugation by V makes sense because by equation (4.3) and (4.6), $2 + b_{21} + b_{41} \neq 0$, and so V is non-singular.

5 Two Bridge Examples

In this section we will examine the deformations of various two bridge knots and links. Two bridge knots and links always admit presentations of a particularly nice form. Given a two bridge knot or link, K, with rational number p/q where q is odd, relatively prime to p and $0 < q < p$ there is always a presentation of the form

$$\pi_1(S^3 \backslash K) = \langle A, B \mid AW = WB \rangle, \tag{5.1}$$

where A and B are meridians of the knot. The word W can be determined explicitly from the rational number, p/q, see [18] for details. We now wish to examine deformations of $\Gamma = \pi_1(S^3 \backslash K)$ that give rise to elements of $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology. Such a deformations must preserve the conjugacy class of both of the meridians of Γ, and so we can assume that throughout the deformation our meridian matrices, A and B, are of the form (4.7). Therefore we can find deformations by solving the matrix

$$AW - WB = 0 \tag{5.2}$$

over $\mathbb{R}$.

5.1 Figure-eight knot

The figure-eight knot has rational number 5/3, and so the word $W = BA^{-1}B^{-1}A$. Solving (5.2) we find that

$$b_{31} = 2,\ a_{34} = \frac{b_{21}}{2(b_{21} - 2)},\ a_{24} = \frac{1}{(b_{21} - 2)},\ a_{14} = \frac{3 - b_{21}}{b_{21} - 2}, \tag{5.3}$$

and so we have found a 1 parameter family of deformations. However, this family does not preserve the conjugacy class of any non-meridional peripheral element. A longitude of the figure-eight is given by $L = BA^{-1}B^{-1}A^2B^{-1}A^{-1}B$, and a simple computation shows that if (5.3) is satisfied then $\operatorname{tr}(L) = \frac{48+(b_{21}-2)^4}{8(b_{21}-2)}$. This curve of representations corresponds to the cohomology classes guaranteed by Theorem 3.2, where γ_1 can be taken to be the longitude. If we want the entire boundary to be parabolic we must add the equation $\operatorname{tr}(L) = 4$ to (5.2) and when we solve we find that over $\mathbb{R}$ the only solution is

$$b_{21} = 4,\ b_{31} = 2,\ a_{14} = -\frac{1}{2},\ a_{34} = 1,\ a_{24} = \frac{1}{2}. \tag{5.4}$$

This proves Theorem 1.1 for the figure-eight knot.

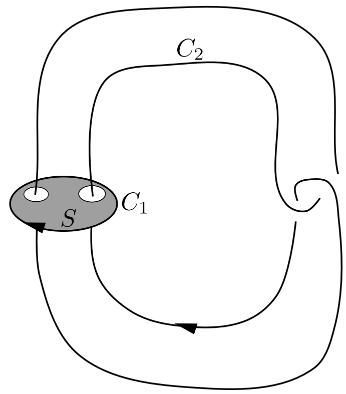

Figure 1: The Whitehead link and its totally geodesic, thrice punctured sphere.

5.2 The Whitehead Link

The Whitehead link has rational number 8/3, and so the word $W = BAB^{-1}A^{-1}B^{-1}AB$. If we again solve (5.2) we see that

$$b_{21} = 4,\ b_{31} = -1,\ a_{14} = 0,\ a_{24} = -2,\ a_{34} = 2, \tag{5.5}$$

and again we see that the Whitehead link has no deformations preserving the conjugacy classes of the boundary elements near the geometric representation, thus proving Theorem 1.1 for the Whitehead link. This time it was not necessary to place any restriction on the trace of a longitude in order to get a unique solution. Theorem 3.2 tells us that there are still 2 dimensions of infinitesimal deformations of Γ that we have not yet accounted for. However, we can find deformations that give rise to these extra cohomology classes. To see this notice that there is a totally geodesic surface that intersects one component of the Whitehead link in a longitude that we can bend along. With reference to Figure 5.2, we denote the totally geodesic surface S, and the two cusps of the Whitehead link by C_1 and C_2. If we bend along this surface the effect on C_1 will be to deform the meridian (or any non-longitudinal, peripheral curve) and leave the longitude fixed. This bending has no effect on C_2, since C_2 intersects S in two oppositely oriented copies of the meridian, and the bending along these two meridians cancel one another (see Theorem 3.3). This picture is symmetric and so there is another totally geodesic surface bounding the longitude of C_2 and the same argument shows that we can find another family of deformations.

Remark 5.1. *Similar computations have been done for the two bridge knots with rational number* 7/3 *and* 9/5 *and in both cases the solutions form a discrete set, however in these two examples it is not known if the cohomology classes coming from Theorem 3.2 are integrable. The details of these computations can be found in* [2] *and serve to complete the proof of Theorem 1.1.*

6 Rigidity and Flexibility After Surgery

In this section we examine the relationship between deformations of a cusped hyperbolic manifold and deformations of manifolds resulting from surgery. The overall idea is as follows: suppose that M is a 1-cusped hyperbolic 3-manifold of finite volume, α is a slope on ∂M, and $M(\alpha)$ is the manifold resulting from surgery along α. If $M(\alpha)$ is hyperbolic with geometric representation ρ_{geo} and ρ_t is a non-trivial family of deformations of ρ_{geo} into $\mathrm{PGL}_4(\mathbb{R})$. Since $\pi_1(M(\alpha))$ is a quotient of $\pi_1(M)$ we can pull ρ_t back to a non-trivial family of representations, $\tilde{\rho}_t$, such that $\tilde{\rho}_t(\alpha) = 1$ for all t (It is important to remember that $\tilde{\rho}_0$ is not the geometric representation for M). In terms of cohomology, we find that the image of the element $\omega \in H^1(M, \mathfrak{sl}(4)_{\tilde{\rho}_0})$ corresponding to $\tilde{\rho}_t$ is trivial in $H^1(\alpha, \mathfrak{sl}(4)_{\tilde{\rho}_0})$. With this in mind we will call slope α-*rigid* if the map $H^1(M, \mathfrak{v}_{\rho_{\text{geo}}}) \to H^1(\alpha, \mathfrak{v}_{\rho_{\text{geo}}})$ is *non-trivial*. Roughly the idea is that if a slope is rigid then we can find deformations that do not infinitesimally fix α. The calculations from section 5.2 show that either meridian of the Whitehead link is a rigid slope.

6.1 The cohomology of ∂M

Before proceeding we need to understand the structure of the cohomology of the boundary. For details of the facts in this section see [13, Section 5]. Let T is a component of ∂M and let γ_1 and γ_2 be generators of $\pi_1(T)$ whose angle is not an integral multiple of $\pi/3$, then we have an injection,

$$H^1(T, \mathfrak{v}_{\rho_0}) \overset{\iota^*_{\gamma_1} \oplus \iota^*_{\gamma_2}}{\to} H^1(\gamma_1, \mathfrak{v}_{\rho_0}) \oplus H^1(\gamma_2, \mathfrak{v}_{\rho_0}), \tag{6.1}$$

where $\iota^*_{\gamma_i}$ is the map induced on cohomology by the inclusion $\gamma_i \hookrightarrow T$. Additionally, if ρ_u is the holonomy of an incomplete, hyperbolic structure, then

$$H^*(\partial M, \mathfrak{v}_{\rho_u}) \cong H^*(\partial M, \mathbb{R}) \otimes \mathfrak{v}^{\rho_u(\pi_1(\partial M))}, \tag{6.2}$$

where $\mathfrak{v}^{\rho_u(\pi_1(\partial M))}$ are the elements of $\mathfrak{v}$ invariant under $\rho_i(\pi_1(\partial M))$. Additionally, for all such representations the image of $H^1(M, \mathfrak{v}_{\rho_u})$ under ι^* in $H^1(\partial M, \mathfrak{v}_{\rho_u})$ is 1 dimensional.

6.2 Deformations coming from symmetries

From the previous section we know that the map $H^1(M, \mathfrak{v}_\rho) \to H^1(\partial M, \mathfrak{v}_\rho)$ has rank 1 whenever ρ is the holonomy of an incomplete, hyperbolic structure, and we would like to know how this image sits inside $H^1(\partial M, \mathfrak{v}_\rho)$. Additionally, we would like to know when infinitesimal deformations coming from cohomology classes can be integrated to actual deformations. Certain symmetries of M can help us to answer this question.

Suppose that M is the complement of a hyperbolic, amphicheiral knot complement, then M admits an orientation reversing symmetry, ϕ, that sends the longitude to itself and the meridian to its inverse. The existence of such a symmetry places strong restrictions on

the shape of the cusp. Let m and l be the meridian and longitude of M, then it is always possible to conjugate so that

$$\rho(m) = \begin{pmatrix} e^{a/2} & 1 \\ 0 & e^{-a/2} \end{pmatrix}, \qquad \rho(l) = \begin{pmatrix} e^{b/2} & \tau_\rho \\ 0 & e^{-b/2} \end{pmatrix}.$$

The value τ_ρ is easily seen to be an invariant of the conjugacy class of ρ and we will henceforth refer to it as the τ *invariant* (see also [6, App B]). When ρ is the geometric representation this coincides with the cusp shape.

Suppose that $[\rho]$ is a representation such that the metric completion of $\mathbb{H}^3/\rho(\pi_1(M))$ is the cone manifold $M(\alpha/0)$ or $M(0/\alpha)$, where $M(\alpha/0)$ (resp. $M(0/\alpha)$) is the cone manifolds obtained by Dehn filling along the meridian (resp. longitude) with a solid torus with singular longitude of cone angle $2\pi/\alpha$. Using the τ invariant we can show that the holonomy of the singular locus of these cone manifolds is a pure translation.

Lemma 6.1. *Let M be a hyperbolic, amphicheiral, knot complement, and let $\alpha \geq 2$. If $M(\alpha/0)$ is hyperbolic then the holonomy of the longitude is a pure translation. Similarly, if $M(0/\alpha)$ is hyperbolic then the holonomy around the meridian is a pure translation.*

Proof. We prove the result for $M(\alpha/0)$. Provided that $M(\alpha/0)$ is hyperbolic and $\alpha \geq 2$, rigidity results for cone manifolds from [14, 16] provides the existence of an element $A_\phi \in \mathrm{PSL}_2(\mathbb{C})$ such that $\rho(\phi(\gamma)) = \overline{A_\phi \rho(\gamma) A_\phi^{-1}}$, where $\overline{\gamma}$ is complex conjugation of the entries of the matrix γ. Thus we see that $\tau_{\rho\circ\phi}$ is $\overline{\tau_\rho}$. On the other hand ϕ preserves l and sends m to its inverse and so we see that the $\tau_{\rho\circ\phi}$ is also equal to $-\tau_\rho$, and so we see that τ_ρ is purely imaginary.

The fact that $\rho(m)$ and $\rho(l)$ commute gives the following relationship between a, b and τ_ρ:

$$\tau_\rho \sinh(a/2) = \sinh(b/2). \tag{6.3}$$

Using the fact that $\mathrm{tr}(\rho(m))/2 = \cosh(a/2)$ and the analogous relationship for $\mathrm{tr}(\rho(l))$ we can rewrite (6.3) as

$$\tau_\rho^2(\mathrm{tr}^2(\rho(m)) - 4) + 4 = \mathrm{tr}^2(\rho(l)). \tag{6.4}$$

For the cone manifold $M(\alpha/0)$, $\rho(m)$ is elliptic and so $\mathrm{tr}^2(\rho(m))$ is real and between 0 and 4. This forces $\mathrm{tr}^2(\rho(l))$ to be real and greater than 4, and we see that $\rho(l)$ is a pure translation. The proof for $M(0/\alpha)$ is identical with the roles of m and l being exchanged. □

With this in mind we can prove the following generalization of [13, Lemma 8.2], which lets us know that ϕ induce maps on cohomology that act as we would expect.

Lemma 6.2. *Let M be the complement of a hyperbolic, amphicheiral knot with geometric representation ρ_{geo}, and let ρ_u be the holonomy of a incomplete, hyperbolic structure whose completion is $M(\alpha/0)$ or $M(0/\alpha)$, where $\alpha \geq 2$, then*

1. *$H^*(\partial M, \mathfrak{v}_{\rho_u}) \cong H^*(\partial M, \mathbb{R}) \otimes \mathfrak{v}^{\rho_u(\pi_1(\partial M))}$ and $\phi_u^* = \phi^* \otimes \mathrm{Id}$, where ϕ_u is the map induced by ϕ on $H^*(\partial M, \mathfrak{v}_{\rho_u})$, ϕ^* is the map induced by ϕ on $H^*(\partial M, \mathbb{R})$, and*

2. $\iota_l^* \circ \phi_0^* = \iota_l^*$ *and* $\iota_m^* \circ \phi_0^* = -\iota_m^*$*, where* ϕ_0^* *is the map induced by* ϕ *on* $H^1(\partial M, \mathfrak{v}_{\rho_{\mathrm{geo}}})$.

Proof. The proof that $H^*(\partial M, \mathfrak{v}_{\rho_u}) \cong H^*(\partial M, \mathbb{R}) \otimes \mathfrak{v}^{\rho_u(\pi_1(\partial M))}$ is found in [13]. For the first part we will prove the result of $M(\alpha/0)$ (the other case can be treated identically). By the previous paragraph we know that $\rho_u(m)$ is elliptic and that $\rho_u(l)$ is a pure translation. Once we have made this observation our proof is identical to the proof given in [13].

For the second part, we begin by observing that by Mostow rigidity there is a matrix $A_0 \in \mathrm{PO}(3,1)$ such that $\rho_0(\phi(\gamma)) = A_0 \cdot \rho_0(\gamma)$, where the action here is by conjugation. The fact that our knot is amphicheiral tells us that the cusp shape of M is imaginary, and so in $\mathrm{PSL}_2(\mathbb{C})$ we can assume that

$$\rho_0(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_0(l) = \begin{pmatrix} 1 & ic \\ 0 & 1 \end{pmatrix},$$

where c is some positive real number. Under the standard embedding of $\mathrm{PSL}_2(\mathbb{C})$ into $\mathrm{SL}_4\mathbb{R}$ (as the copy of $\mathrm{SO}(3,1)$ that preserves the form $x_1^2 + x_2^2 + x_3^2 - x_4^2$) we see that

$$\rho_0(m) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \rho_0(l) = \exp\begin{pmatrix} 0 & 0 & -c & c \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}.$$

Thus we see that $A_0 = T\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where T is some parabolic isometry that fixing the vector $(0,0,1,1)$, which corresponds to ∞ in the upper half space model of $\mathbb{H}^3$. Such a T will be of the form

$$T = \exp\begin{pmatrix} 0 & 0 & -a & a \\ 0 & 0 & -b & b \\ a & b & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix},$$

where a and b can be thought of as the real and imaginary parts of the complex number that determines this parabolic translation. Since the cusp shape is imaginary the angle between m and l is $\pi/2$ and we know from [13, Lemma 5.5] that the cohomology classes given by the cocycles z_m and z_l generate $H^1(\partial M, \mathfrak{v}_{\rho_0})$. Here z_m is given by $z_m(l) = 0$ and $z_m(m) = a_l$, where

$$a_l = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and z_l given by $z_l(m) = 0$ and $z_l(l) = a_m$, where

$$a_m = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Observe that $\phi_0^* z_l(m) = 0 = z_l(m)$ and that

$$\phi_0^* z_m(m) = A_0^{-1} \cdot z_m(m^{-1}) = -A_0^{-1} \cdot \rho_0(m^{-1}) \cdot a_l.$$

In [13] it is shown how the cup product associated to the Killing form on $\mathfrak{v}$ gives rise to a non-degenerate pairing. We can now use this cup product to see that $\iota_m^* \phi_0^* z_m(m)$ and $-\iota_m^* z_m(m)$ are cohomologous. The cup product yields a map $H^1(m, \mathfrak{v}_{\rho_0}) \otimes H^0(m, \mathfrak{v}_{\rho_0}) \to H^1(m, \mathbb{R}) \cong \mathbb{R}$ given by $(a \cup b)(m) = B(a(m), b) = 8\mathrm{tr}\,(a(m)b)$ (where we are thinking of $H^1(m, \mathbb{R})$ as homomorphisms from $\mathbb{Z}$ to $\mathbb{R}$). Observe that

$$(\iota_m^* \phi_0^* z_m \cup a_m)(m) = B(\phi_0^* z_m(m), a_m) = 32 = -B(a_l, a_m) = -(\iota_m^* z_m \cup a_m)(m).$$

Since this pairing is non-degenerate and since $[\iota_m^* \phi_0^* z_m]$ and $[-\iota_m^* z_m]$ both live in the same 1 dimensional vector space we see that they must be equal. Thus we see that $\iota_m^* \circ \phi_0^* = -\iota_m^*$. After observing that

$$(\iota_l^* \phi_0^* z_l \cup a_l)(l) = B(\phi_0^* z_l(l), a_l) = -32 = B(a_m, a_l) = (\iota_l^* z_l \cup l_l)(l),$$

a similar argument shows that $\iota_l^* \circ \phi_0^* = \iota_l^*$

□

This lemma immediately helps us to answer the question of how the image of $H^1(M, \mathfrak{v}_{\rho_0})$ sits inside of $H^1(\partial M, \mathfrak{v}_{\rho_0})$. Since $\pi_1(\partial M)$ is invariant under ϕ we see that the image of $H^1(M, \mathfrak{v}_{\rho_0})$ is invariant under the involution ϕ^*, and so the image will be either the ± 1 eigenspace of ϕ^*. In light of Lemma 6.2 we see that these eigenspaces sit inside of $H^1(l, \mathfrak{v}_{\rho_0})$ and $H^1(m, \mathfrak{v}_{\rho_0})$, respectively. Under the hypotheses of Theorem 1.3 the previous fact is enough to show that certain infinitesimal deformations are integrable.

Proof of Theorem 1.3. In this proof $O = M(n/0)$ and N will denote a regular neighborhood of the singular locus of $M(n/0)$. Hence we can realize O as $M \sqcup N$. By using a Taylor expansion we see that given a family of representations ρ_t of $\pi_1(O)$ into $\mathrm{SL}_4(\mathbb{R})$ such that ρ_0 is the geometric representation, ρ_{geo}, of O. We can therefore write

$$\rho_t(\gamma) = (I + u_1(\gamma)t + u_2(\gamma)t^2 + \ldots)\rho_{\mathrm{geo}}(\gamma),$$

where the u_i are 1-cochains. The ρ_t will be homomorphisms if and only if for each k

$$\delta u_k + \sum_{i=1}^{k} u_i \cup u_{k-i} = 0, \tag{6.5}$$

where $a \cup b$ is the 2-cochain given by $(a \cup b)(c, d) = a(c)c \cdot b(d)$, where the action is by conjugation. Conversely, if we are given a cocycle u_1 and a collection $\{u_k\}$ such that (6.5) is satisfied for each k we can apply a deep theorem of Artin [1] to show that we can find an actual deformation ρ_t in $\mathrm{PGL}_4(\mathbb{R})$ that is infinitesimally equal to u_1. Using Weil rigidity and the splitting (3.5) we see that $H^1(M(n/0), \mathfrak{sl}(4)_{\rho_n}) \cong H^1(M(n/0), \mathfrak{v}_{\rho_n})$, where ρ_n is the holonomy of the incomplete structure whose completion is $M(n/0)$. Therefore, given a cohomology class $[u_1]$, we can assume that $[u_1] \in H^1(M(n/0), \mathfrak{v}_{\rho_n})$, and we need to find cochains u_k satisfying (6.5). In order to simplify notation $H^1(*, \mathfrak{v}_{\rho_n})$ will be denoted $H^1(*)$. The orbifold O is finitely covered by an aspherical manifold, and so by combining a transfer argument [13] with the fact that singular cohomology and group cohomology coincide for aspherical manifolds [27], we can conclude that group cohomology for $\pi_1(O)$ is the same as singular cohomology for O. Therefore we can use a Mayer-Vietoris sequence to analyze cohomology. Consider the following section of the sequence.

$$H^0(M) \oplus H^0(N) \to H^0(\partial M) \to H^1(O) \to H^1(M) \oplus H^1(N) \to H^1(\partial M). \tag{6.6}$$

Next, we will determine the cohomology of N. Since N has the homotopy type of S^1 it will only have cohomology in dimension 0 and 1. Since $\rho_n(\partial M) = \rho_n(N)$ we see that $H^0(N) \cong H^0(\partial M)$ (both are 1-dimensional). Finally by duality we see that $H^1(N)$ is also 1-dimensional. Additionally, $H^0(O)$ is trivial since ρ_n is irreducible. Combining these facts, we see that the first arrow is an isomorphism and thus the penultimate arrow of (6.6) is injective. We also learn that $H^1(O)$ injects into $H^1(M)$, because if a cohomology class from $H^1(O)$ dies in $H^1(M)$ then exactness tells us that it must also die when mapped into $H^1(N)$ (since $H^1(N)$ injects into $H^1(\partial M)$). However, this contradicts the fact that $H^1(O)$ injects into $H^1(M) \oplus H^1(N)$. Using the fact that the longitude is a rigid slope, Lemma 6.2, and [13, Cor 6.6] we see that ϕ^* acts as the identity on $H^1(O)$, but since $H^1(M)$ and $H^1(N)$ have the 1-eigenspace of ϕ^* as their image in $H^1(\partial M)$, the last arrow of (6.6) is not a surjection, and so $H^1(O)$ is 1-dimensional.

Duality tells us that $H^2(O)$ is 1-dimensional and we will now show that ϕ^* act as multiplication by -1. By duality we see that $H^3(O) = 0$ and so the Mayer-Vietoris sequence contains the following piece.

$$H^1(M) \oplus H^1(N) \to H^1(\partial M) \to H^2(O) \to H^2(M) \oplus H^2(N) \to H^2(\partial M) \to 0. \tag{6.7}$$

Since $H^2(O)$ is 1-dimensional, the second arrow of (6.7) is either trivial or surjective. If this arrow is trivial then the third arrow is an injection and thus an isomorphism for dimensional reasons, but this is a contradiction since the penultimate arrow is a surjection and $H^2(\partial M)$ is non-trivial. Since the first arrow of (6.7) has the 1-eigenspace of ϕ^* as its image we see that the -1-eigenspace of ϕ^* surjects $H^2(O)$. However, since the Mayer-Vietoris sequence is natural and ϕ respects the splitting of O into $M \cup N$ we see that ϕ^* acts on $H^2(O)$ as -1.

We can now construct the u_i. Let $[u_1]$ be a generator of $H^1(O)$ (we know this is one dimensional by duality). Since ϕ is an isometry of a finite volume hyperbolic manifold we know that it has finite order when viewed as an element of $\mathrm{Out}(\pi_1(\mathrm{M}))$ [25], and so there exists a finite order map ψ that is conjugate to ϕ. Because the maps ϕ and ψ are conjugate, they act in the same way on cohomology [7]. Let L be the order of ψ. Since ψ acts as the identity on $H^1(O)$, the cocycle

$$u_1^* = \frac{1}{L}(u_1 + \phi(u_1) + \ldots + \phi^{L-1}(u_i))$$

is both invariant under ψ and cohomologous to u_1. By replacing u_1 with u_1^* we can assume that u_1 is invariant under ψ. Next observe that

$$-u_1 \cup u_1 \sim \psi(u_1 \cup u_1) = \psi(u_1) \cup \psi(u_1) = u_1 \cup u_1,$$

and so we see that $u_1 \cup u_1$ is cohomologous to 0, and so there exists a 1-cochain, u_2, such that $\delta u_2 + u_1 \cup u_1 = 0$. Using the same averaging trick as before we can replace u_2 by the ψ-invariant cochain u_2^*. By invariance of u_1 we see that u_2^* has the same boundary as u_2 so this replacement does not affect the first part of our construction. Again we see that

$$-(u_1 \cup u_2 + u_2 \cup u_1) \sim \psi(u_1 \cup u_2 + u_2 \cup u_1) = u_1 \cup u_2 + u_2 \cup u_1,$$

and so there exist u_3 such that (6.5) is satisfied. Repeating this process indefinitely, we can find the sequence u_i satisfying (6.5), and thus by Artin's theorem we can find our desired deformation.

□

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